# The wall jet

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SUMMARY

This paper considers the flow due to a jet spreading out over a plane surface, either radially or in two dimensions. Solutions of the boundary layer equations are sought, according to which the form of the velocity distribution across the jet does not vary along its length. For laminar flow, such a similarity solution is obtained explicitly. For turbulent flow, an eddy viscosity is introduced, and it is eventually seen that complete similarity is not attainable, but that confident predictions can nevertheless be made about the nature of the velocity distribution and the rate of growth of the wall jet.

## 1. INTRODUCTION

When a jet of air strikes a surface at right angles and spreads out radially over it, it forms what will be termed a wall jet. Such a flow is produced by the downwards-directed jet from a vertical-take-off aircraft spreading out over the ground, or by a jet of water from a tap falling into a partly full sink and spreading out over the bottom. It should be noted that the flow due to a jet of water falling into an empty sink is in a quite different category, as then the condition at the free surface is that the pressure is constant; for a wall jet, as for a free jet, the corresponding condition is that the radial velocity component falls to zero at the outer edge of the jet.

Plane wall jets may also arise. If the water levels are different in two sections of a canal, separated by a sluice, and if the sluice is slightly raised, the flow into the section with the lower water level will take the form of a wall jet. Alternatively, if a plane jet impinges on a fixed plate parallel to the direction of flow, a wall jet will be produced on each side of the plate.

The theory of such wall jets, radial or plane, laminar or turbulent, forms the subject of this paper. It does not appear to have received attention previously. The appropriate boundary layer equations are set up, and a search is made for a similarity solution, in which the form of the velocity distribution across the jet does not vary along its length. This is a familiar procedure in boundary layer theory. Two similarity exponents, giving the variation with distance of the maximum velocity and the jet width, have to be determined, and one relation between them is obtained from the boundary layer equations themselves. In the corresponding problem for a free jet, a second relation is deduced from the constancy of momentum flux, and for a boundary layer the second relation is even simpler in form, since the velocity must vary in the same manner as does the known external flow. For a wall jet, neither of the principles can be applied. It is shown in § 2 that the solution depends on an eigenvalue problem, which for laminar flow is satisfactorily solved in §§ 2 and 3. The calculation of the jet profile is carried out in § 4, and it is found possible to perform the integrations analytically. The same velocity distribution applies for both radial and plane wall jets, a fortunate circumstance which occurs in all the cases considered in this paper, and greatly shortens the work.

In most practical examples, the wall jet will be turbulent. The attempts in §§ 5–9 at predicting its form are all based on the idea of replacing the molecular viscosity by an effective eddy viscosity, and requiring this to vary in a plausible manner. The first attempt, in § 5, makes use of the hypothesis for free turbulent flow due to Prandtl (1942), according to which the eddy viscosity is constant across the breadth of the jet. The velocity profile thus derived is shown to be identical with that in the laminar case, and further implications of this result are noted. Near the wall, this solution is not in accord with preliminary experiments (Bakke 1956), and in § 6 a second attempt is made, with an eddy viscosity as required to satisfy the law due to Blasius (1913) for flow in a pipe. The solution again is calculated without difficulty, and might be expected to be accurate in the region near the wall. A more realistic procedure is to postulate an eddy viscosity based on Blasius's law near the wall and Prandtl's hypothesis further out. This implies that complete similarity is no longer possible. Also, the eigenvalue problem to determine the similarity exponents, which in the earlier attempts was solved by a simple extension of the analysis for a laminar wall jet, has to be considered afresh. In spite of these difficulties, in §§ 7–9 an analysis is carried out which results in definite predictions as to the velocity distribution and the values of the similarity exponents, as slowly varying functions of the Reynolds number. As in all problems involving Prandtl's hypothesis, the value of a single constant has to be assigned from experimental observations. The final results are put forward with some confidence in the belief that they do correctly predict the main features of an actual turbulent wall jet.

In what follows, the theory is developed for the case of a radial jet, as being of the greater practical interest. The modifications required to make the analysis applicable to a plane wall jet are only slight, and are noted at the end of each section.

## 2. Similarity solutions

We now consider the equations governing a radial laminar jet flowing over a plane wall. On the boundary layer approximation, the pressure is everywhere uniform, and the momentum equation is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{2.1}$$

where $x$ and $y$ denote distances along and normal to the wall, $x$ being measured from the jet axis, $u$ and $v$ the corresponding velocity components, and $\nu$ the kinematic viscosity. The equation of continuity is

$$\frac{\partial(xu)}{\partial x} + \frac{\partial(xv)}{\partial y} = 0, \tag{2.2}$$

and hence there is a Stokes stream function $\psi$ satisfying

$$xu = \frac{\partial\psi}{\partial y}, \qquad xv = -\frac{\partial\psi}{\partial x}. \tag{2.3}$$

The boundary conditions are

$$u = v = 0 \quad \text{at } y=0, \qquad u \to 0 \quad \text{as } y \to \infty. \tag{2.4}$$

Consider the possibility that there shall be a similarity solution of these equations, with $u \propto x^a$, and the jet thickness $\delta \propto x^b$. The two sides of (2.1) will vary with $x$ in the same manner if

$$a + 2b = 1. \tag{2.5}$$

In other boundary layer problems, a second relation between $a$ and $b$ is obtained from the general nature of the flow. Thus for a boundary layer in a stream $U = cx^m$, it is at once clear that $a = m$, and for a free jet, constancy of momentum flux gives the second relation. For a wall jet, there is no help from the boundary conditions, momentum is not conserved, and there is no obvious quantity to consider instead. Indeed $\psi \equiv 0$ satisfies both equations and boundary conditions. This suggests that only for particular values of $a$ and $b$ will non-trivial solutions arise.

For convenience, we introduce non-dimensional variables by writing

$$u = U\bar{u}, \quad v = U\bar{v}, \quad x = \nu\bar{x}/U, \quad y = \nu\bar{y}/U, \quad \psi = \nu^2\bar{\psi}/U, \tag{2.6}$$

where $U$ is a constant velocity. Using (2.5), we write

$$\bar{\psi} = \bar{x}^{2-b}f(\eta),$$

where

$$\eta = (2-b)\bar{y}\bar{x}^{-b}.$$

Then

$$\bar{u} = (2-b)\bar{x}^{1-2b}f'(\eta),$$

and equation (2.1) becomes

$$f''' + ff'' + \alpha f'^2 = 0, \tag{2.7}$$

where

$$\alpha = (2b-1)/(2-b). \tag{2.8}$$

The boundary conditions (2.4) require that

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 0. \tag{2.9}$$

For any $\alpha$, there are solutions of (2.7) with $f'(\infty) = 0$. We now investigate the possibility that, for such a solution, $f$ and $f'$ shall both be zero at some finite $\eta$, so that (2.9) may be satisfied, with a suitably chosen zero of $\eta$. Integrating (2.7) between the limits $\eta$ and $\infty$, we obtain

$$f'' + ff' - (\alpha-1)g = 0, \tag{2.10}$$

where

$$g(\eta) = \int_\eta^\infty f'^2\, d\eta.$$

Note that $g$ is certainly positive. Multiplying (2.10) by $f'$ and integrating again, we have

$$\tfrac{1}{2}f'^2 - fg + (\alpha-2)\int_\eta^\infty f'g\,d\eta = 0. \tag{2.11}$$

If, at the first zero of $f'$ to be encountered as $\eta$ decreases, $f$ is also zero, then since the integral in (2.11) is certainly positive, this equation requires that $\alpha = 2$. Equations (2.5) and (2.8) now show that

$$a = -\tfrac{3}{2}, \qquad b = \tfrac{5}{4}. \tag{2.12}$$

Unique values have thus been found for the similarity exponents. The only assumption in the argument, that $f'$ is nowhere negative, i.e. that there is no reversed flow, is clearly acceptable on physical grounds.

For a plane wall jet, the equation of continuity $\partial u/\partial x + \partial v/\partial y = 0$ is satisfied by introducing a stream function $\psi$ such that $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$. On writing $\bar{\psi} = \psi/\nu = \bar{x}^{1-b}f(\eta)$, where $\eta = (1-b)\bar{x}^{-b}$, we obtain (2.7) precisely as before, where now $\alpha = (2b-1)/(1-b)$. The proof that $\alpha = 2$ is unchanged, and hence, for a plane wall jet,

$$a = -\tfrac{1}{2}, \qquad b = \tfrac{3}{4}. \tag{2.13}$$

The equations being identical, the velocity distributions across the jet are the same in the radial and plane cases.

Before proceeding further, we may note some important implications of (2.10). If $f$ and $f''$ are to vanish simultaneously, with $f' > 0$, this equation shows that $\alpha = 1$. These are precisely the conditions which hold for a free jet (where $\partial u/\partial y = 0$ on the centre line $y = 0$), and (2.7) with $\alpha = 1$ is indeed the equation for the velocity distribution in a plane free jet, found by Schlichting (1933) and Bickley (1937). The above arguments show that this same velocity distribution applies also to the similar state of a radial free jet, as would occur when a radial wall jet had passed beyond the edge of a finite circular plate. For here, too, the equation must be of the same form as (2.7). The similarity exponents are found from (2.8) to be $a = -1$, $b = 1$. These results for a radial free jet are in agreement with those of Squire (1955). A practical example of a radial free jet occurs, approximately, in the circular grilling burner of a gas cooker. The fact that, for a free jet, (2.10) leads to the same values of the similarity exponents as does a consideration of momentum flux, suggests that it may, after all, be possible to deduce the values for a wall jet by analogous considerations. This is indeed the case, as will be shown in the next section.

## 3. Exterior momentum flux

Our aim is to deduce an integral relation for a radial laminar wall jet, from the basic equations (2.1) to (2.4). Multiply (2.1) by $x$ and integrate with respect to $y$ between the limits $y$ and $\infty$, using the condition that $u \to 0$ as $y \to \infty$. By (2.2),

$$\int_y^\infty xv\frac{\partial u}{\partial y}\,dy = -xvu + \int_y^\infty u\frac{\partial(xu)}{\partial x}\,dy.$$

and so we obtain

$$\frac{\partial}{\partial x}\int_y^\infty xu^2\,dy - xvu + \nu x\frac{\partial u}{\partial y} = 0. \tag{3.1}$$

Multiplying by $xu$, and integrating with respect to $y$ between the limits 0 and $\infty$, we then have

$$\frac{\partial}{\partial x}\int_0^\infty xu\left\{\int_y^\infty xu^2\,dy\right\}dy - \int_0^\infty \frac{\partial(xu)}{\partial x}\left\{\int_y^\infty xu^2\,dy\right\}dy - \\ - \int_0^\infty x^2\nu u^2\,dy + \left[\tfrac{1}{2}\nu x^2u^2\right]_0^\infty = 0. \tag{3.2}$$

From the continuity equation, the second term of (3.2) is

$$\left[xv\int_y^\infty xu^2\,dy\right]_0^\infty + \int_0^\infty x^2vu^2\,dy.$$

Now at $y = 0$, $u = v = 0$, and so (3.2) reduces to

$$\frac{\partial}{\partial x}\int_0^\infty xu\left\{\int_y^\infty xu^2\,dy\right\}dy = 0,$$

or

$$F = \int_0^\infty xu\left\{\int_y^\infty xu^2\,dy\right\}dy = \text{constant}. \tag{3.3}$$

Physically, this equation may perhaps be interpreted as saying that the flux of exterior momentum flux is constant, but this is hardly a familiar concept. The close similarity between the types of expression in equations (2.11) and (3.2) may be noted. In the case of a similarity solution, (3.3) shows at once that

$$3a + 2b + 2 = 0, \tag{3.4}$$

and, taken together with (2.5), this leads to the values of $a$ and $b$ already given in (2.12). If there is reversed flow, it is possible that $F$ is zero, and in this case the deduction (3.4) cannot be made. So here, as in § 2, there is the mathematical possibility of solutions with reversed flow, for some other values of the similarity exponents.

The arguments leading to the constancy of $F$ are not dependent on the velocity profiles being similar. The similarity solution requires an infinite velocity and zero jet width at $x = 0$, so in an actual wall jet there must be a region where the velocity distribution differs from its final similar state. Equation (3.3) gives valuable information about the state of affairs in this region.

We may note that a rough estimate of the magnitude of $F$ may be deduced from a knowledge of conditions in the impinging free jet, by writing

$$\begin{aligned} F = \int_0^\infty xu\left\{\int_y^\infty xu^2\,dy\right\}dy &= u^*\int_0^\infty xu\left\{\int_y^\infty xu\,dy\right\}dy \\ &= \tfrac{1}{2}u^*\int_0^\infty -\frac{\partial}{\partial y}\left\{\int_y^\infty xu\,dy\right\}^2 dy \\ &= \tfrac{1}{2}u^*\left\{\int_0^\infty xu\,dy\right\}^2, \end{aligned} \tag{3.5}$$

where $u^*$ is a velocity typical of that in the jet. Thus

$$F = \tfrac{1}{2}(\text{typical velocity}) \times (\text{volume flow per radian})^2.$$

Neither the volume flow nor the magnitude of the fluid velocity will alter greatly when the jet is deflected on striking the wall, so that the value of (3.5), estimated from conditions in the impinging jet, provides a measure of $F$ in the wall jet.

For plane flow, the analysis of this section is unchanged, except that the $x$'s are omitted and the final result,

$$F = \int_0^\infty u\left\{\int_y^\infty u^2\,dy\right\}dy = \text{constant}, \tag{3.6}$$

leads to

$$3a + 2b = 0, \tag{3.7}$$

again agreeing with the previously obtained result.

If the flow is turbulent, momentum flux arguments still hold for a free jet, but for a wall jet $F$ will remain constant only under certain special assumptions about the nature of the turbulent stresses. These matters will be discussed in later sections.

## 4. Velocity distribution

The expression for the stream function given in §2, with the value $b = \frac{5}{4}$ found in (2.12), was $\psi = (\nu^5 x^3/U)^{1/4} f(\eta)$, where $\eta = \frac{3}{4}(\nu^3/Ux^5)^{1/4} y$, and hence $u = \frac{3}{4}(\nu^3/Ux^3)^{1/2} f'(\eta)$. The equation for $f$ was shown to be

$$f''' + ff'' + 2f'^2 = 0 \tag{4.1}$$

with boundary conditions $f(0) = f'(0) = 0$, $f'(\infty) = 0$.

Multiply the equation by $f$. The first term may now be integrated by parts, and the remaining two form a perfect differential. Hence

$$ff'' - \tfrac{1}{2}f'^2 + f^2 f' = 0, \tag{4.2}$$

since $f'(\infty) = 0$. Multiply by $f^{-3/2}$, and integrate again. Thus

$$f^{-1/2}f' + \tfrac{2}{3}f^{3/2} = \text{constant}. \tag{4.3}$$

Now if $f_0(\eta)$ is a solution of (4.1), so also is $f_1(\eta) = Af_0(A\eta)$, for any constant $A$, and $f_1$ satisfies the boundary conditions if $f_0$ does. But the effect of selecting $f_1$ instead of $f_0$ as the solution for substitution into the expressions given above for $\psi$, $\eta$ and $u$ is precisely the same as that of changing the value of the arbitrary constant velocity $U$ to $U/A^4$. Hence, without loss of generality, we may select the solution with $f(\infty) = 1$, and take the constant in (4.3) to have the value $\frac{2}{3}$. Write $f = g^2$; then $f' = 2gg'$, and (4.3) becomes

$$g' = \tfrac{1}{3}(1 - g^3), \tag{4.4}$$

which gives on integration

$$\eta = \log\frac{\sqrt{(1+g+g^2)}}{1-g} + \sqrt{3}\tan^{-1}\frac{\sqrt{3}g}{2+g}. \tag{4.5}$$

The values of $\eta$, $f$ and $f'$ corresponding to given values of $g$ may now be tabulated. The variations with $\eta$ of $f$ and $f'$ are illustrated in figure 1. The main properties of the velocity distribution are easily calculated directly. Thus, the shearing stress at the wall is related to $f''(0) = \frac{2}{9}$, and the maximum value of $f'$ is $2^{-5/3} \doteqdot 0{\cdot}315$, occurring at the point at which $f = 2^{-4/3} \doteqdot 0{\cdot}397$.

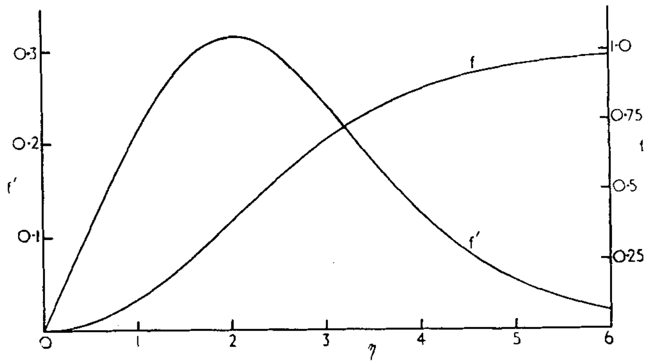


Figure 1. Laminar wall jet. Variation of mass flux ($f$) and velocity ($f'$) with distance from the wall ($\eta$).

The arbitrary velocity $U$ may be eliminated by introducing the quantity $F$ of § 3. It is easily verified that

$$F = \frac{3\nu^4}{4U}\int_0^\infty f'\left\{\int_\eta^\infty f'^2\,d\eta\right\}d\eta = \frac{3\nu^4}{40U}. \tag{4.6}$$

Hence we may write

$$\left.\begin{aligned} \psi &= \left(\frac{40F\nu x^3}{3}\right)^{1/4} f(\eta), \\ u &= \left(\frac{15F}{2\nu x^3}\right)^{1/2} f'(\eta), \\ \eta &= \left(\frac{135F}{32\nu^3 x^5}\right)^{1/4} y. \end{aligned}\right\} \tag{4.7}$$

The skin-friction $\tau_0$ is given by

$$\frac{\tau_0}{\rho} = \nu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \left(\frac{125F^3}{216\nu x^{11}}\right)^{1/4}. \tag{4.8}$$

For a plane wall jet, the expressions corresponding to (4.6), (4.7) and (4.8) are

$$\left.\begin{aligned} F &= \frac{\nu^2 U}{40}, \\ \psi &= (40F\nu x)^{1/4} f(\eta), \\ u &= \left(\frac{5F}{2\nu x}\right)^{1/2} f'(\eta), \\ \eta &= \left(\frac{5F}{32\nu^3 x^3}\right)^{1/4} y, \\ \frac{\tau_0}{\rho} &= \frac{1}{9}\left(\frac{125F^3}{8\nu x^5}\right)^{1/4}. \end{aligned}\right\} \tag{4.9}$$

## 5. Prandtl's hypothesis

In most practical situations, the wall jet will be turbulent, and we now investigate how the foregoing analysis must be modified to apply in this case. If we introduce the concept of an eddy viscosity $\epsilon$, the boundary layer equation (2.1) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\epsilon\frac{\partial u}{\partial y}\right), \tag{5.1}$$

where $u$ and $v$ now denote the components of the mean velocity. The continuity equation (2.2) and the boundary conditions (2.4) are unchanged.

Some assumption must be made about the behaviour of $\epsilon$. The simplest and most convenient one, which has proved satisfactory in describing free turbulent boundary layer flows, is the hypothesis due to Prandtl (1942), that the eddy viscosity is constant across the layer, and is proportional to the product of the maximum mean velocity and the width of the layer. A wall jet would appear to be half-way between a free flow and a wall flow, so it is not unreasonable to adopt Prandtl's hypothesis as a first attempt.

If there is to be a similarity solution of (5.1), with $u \propto x^a$ and $\delta \propto x^b$, we then require that $\epsilon \propto x^{a+b}$, and the two sides of (5.1) will vary with $x$ in the same manner if

$$b = 1. \tag{5.2}$$

In the arguments of § 3 relating to exterior momentum flux, $\nu$ has to be replaced by $\epsilon$, but since this is a function of $x$ only, the analysis applies unchanged. Once again, $F =$ constant, whether or not there is similarity. For a similarity solution, (3.4) holds as before, showing that

$$a = -\tfrac{4}{3}. \tag{5.3}$$

To study the velocity distribution, introduce the non-dimensional variables (2.6), and let

$$\epsilon = \lambda \bar{x}^{-1/3}\nu, \tag{5.4}$$

where $\lambda$ is a constant, to be determined later. Then if we write

$$\psi = \bar{x}^{2/3} f(\eta), \qquad \eta = \frac{2\bar{y}}{3\lambda\bar{x}} = \frac{2y}{3\lambda x}, \tag{5.5}$$

(5.1) shows that $f$ satisfies $f''' + ff'' + 2f'^2 = 0$, with boundary conditions $f(0) = f'(0) = 0, f'(\infty) = 0$, precisely as for the laminar case. The solution of § 4 is still applicable, and thus, on Prandtl's hypothesis, the velocity profile is identical in the laminar and turbulent cases. This result follows at once from the integral arguments of § 2, as soon as it is established that the equation is of the same basic type in the two cases, and it applies equally well to a plane turbulent wall jet.

For a plane free jet, it was shown by Görtler (1942) that, on the basis of Prandtl's hypothesis, the velocity profile of a turbulent jet is the same as that of a laminar one. This again is an immediate deduction from the arguments of § 2. Further, a radial free jet, as described in § 2, would also have this same velocity profile in either laminar or turbulent flow. For all these jets, the use of Prandtl's hypothesis is well justified.

It would indeed be satisfactory if this simple approach sufficed to describe a turbulent wall jet. However, it is found by experiment (Bakke 1956) that the velocity gradient near the wall is much greater than in the velocity distribution of figure 1. Consequently, we shall not proceed to set out further details of the solution based on Prandtl's hypothesis. We merely note that, in this method, the only undetermined quantity is the constant $\lambda$ introduced in (5.4), and this would be chosen to give the best agreement with experiment, as in other problems where Prandtl's hypothesis is used.

## 6. Effect of the wall

Once it is conceded that there is large velocity gradient near the wall, the idea of a free turbulent flow is seen to be an inadequate one. It becomes clear that the friction at the wall is a decisive agency, and must be given a prominent place in any theory. We shall retain the concept of an eddy viscosity, but shall require it to vary in a suitable manner across the breadth of the wall jet.

We may make use of the empirical formula due to Blasius (1913), based on a study of turbulent pipe flow, in the form

$$\tau_0 = 0{\cdot}0225\rho U^2\left(\frac{\nu}{Ua}\right)^{1/4}, \tag{6.1}$$

where $U$ is the maximum velocity and $a$ the radius of the pipe. The formula is often written with the mean velocity and the pipe diameter in place of $U$ and $a$; this merely involves a modification of the numerical coefficient. Since $\tau_0$ is governed by conditions near the wall, it has been argued that $U$ and $a$ in (6.1) may be replaced by $u$ and $y$, so that the equation becomes

$$\tau_0 = 0{\cdot}0225\rho u^2\left(\frac{\nu}{uy}\right)^{1/4}. \tag{6.2}$$

This may be expected to hold near the wall in any turbulent boundary layer flow, outside the viscous sublayer. A discussion of equation (6.2), first discovered by Prandtl, is given by Schlichting (1955, p. 404). He shows that the formula is adequate at Reynolds numbers up to $10^5$. This power law expression is easier to apply in our present theory than is the logarithmic relation which is theoretically more acceptable, and its accuracy should be quite adequate for our purposes. At higher Reynolds numbers a formula similar to (6.2), with a suitably modified exponent, might be used.

Equation (6.2) implies that, near the wall, $u \propto y^{1/7}$. The shearing stress $\tau = \epsilon\, \partial u/\partial y$ has a finite non-zero value at the wall and hence, for fixed $x$,

$$\epsilon \propto y^{6/7} \propto u^6. \tag{6.3}$$

The assumption we shall now make is that $\epsilon$ is proportional to $u^6$, in accordance with (6.3). At first we shall assume that this relation holds throughout the whole width of the jet, though later we shall apply it only in a limited region near the wall. Thus, we choose

$$\epsilon = m(x)u^6, \tag{6.4}$$

where $m$ is as yet unspecified. The main reason for choosing $\epsilon$ to be a function of $u$ rather than of $y$, is that in consequence $\eta$ does not appear explicitly in the equations, but the behaviour is also satisfactory in that $\epsilon$ will not continue to rise beyond the velocity maximum. An additional and rather amusing point is that the exterior momentum flux arguments of § 3 still apply in this case. The last term of equation (3.1) becomes $mxu^6\, \partial u/\partial y$, and hence the last term of (3.2) is replaced by $\frac{1}{8}mx^2u^8$, which vanishes at the wall and at the outer edge of the jet. Consequently equations (3.3) and (3.4) remain true. However, it is not really plausible to consider that this form for $\epsilon$ is applicable in the outer parts of the jet, where the velocity is falling to zero again.

Consider the conditions for a similarity solution of the boundary layer equation (5.1), in which $u \propto x^a$, $\delta \propto x^b$. Blasius's formula (6.2) requires that $\tau_0 \propto u^{7/4}\delta^{-1/4}$, and hence

$$\epsilon \propto u^{3/4}\delta^{3/4} \propto x^{3(a+b)/4}. \tag{6.5}$$

The two sides of (5.1) will depend on $x$ in the same manner if

$$a+5b = 4. \tag{6.6}$$

Introduce the non-dimensional variables (2.6) and, in accordance with the above results, let

$$\bar{\psi} = \bar{x}^{5-4b}f(\eta), \qquad \eta = \frac{5-4b}{\lambda}\bar{y}\bar{x}^{-b}, \tag{6.7}$$

and

$$\epsilon = A\lambda\bar{x}^{3-3b}f'^6\nu, \tag{6.8}$$

where $\lambda$ and $A$ are constants. The reasons for the introduction of the second constant will appear in the next section. Equation (5.1) then gives

$$\frac{d}{d\eta}(Af'^6f'')+ff''+\alpha f'^2 = 0, \tag{6.9}$$

where

$$\alpha = (5b-4)/(5-4b). \tag{6.10}$$

The boundary conditions are $f' \propto \eta^{1/7}$ as $\eta \to 0$, $f'(\eta_\infty) = 0$, where $\eta_\infty$ is the value of $\eta$ at the outer edge of the jet. As a result of the peculiar choice of $\epsilon$, $\eta_\infty$ is finite, as will be seen later. From (6.5) and (6.10), we have

$$a = -\frac{9\alpha}{5+4\alpha}, \qquad b = \frac{4+5\alpha}{5+4\alpha}. \tag{6.11}$$

The integral arguments of § 2 may be applied as before to show that $\alpha = 2$, a result which also follows from the considerations of exterior momentum flux given above. With this value, (6.11) shows that

$$a = -1{\cdot}38, \qquad b = 1{\cdot}08, \tag{6.12}$$

and equation (6.9) may be integrated directly. By a suitable choice of $\lambda$, we may take $A = 1$, and the equation then becomes

$$\frac{d}{d\eta}(f'^6 f'') + ff'' + 2f'^2 = 0. \tag{6.13}$$

Multiplying by $f$ and integrating, we obtain

$$ff'^6 f'' - \tfrac{1}{8} f'^8 + f^2 f' = 0, \tag{6.14}$$

the constant being zero in view of the required behaviour of $f$ near $\eta = 0$. Multiply (6.14) by $7f^{-15/8}$, and integrate again,

$$f^{-7/8} f'^7 + \tfrac{56}{9} f^{9/8} = \text{constant}. \tag{6.15}$$

Choose the constant so that $f = 1$ when $f' = 0$. Solutions for other values of the constant are then expressible as $X^7 f(X^{-5}\eta)$, for some constant $X$. We then have

$$\tfrac{9}{56} f'^7 = f^{7/8} - f^2. \tag{6.16}$$

It is convenient to write $g(\xi) = f^{7/8}$, where $\xi = \frac{7}{8}(\frac{56}{9})^{1/7}\eta$. Equation (6.15) then shows that

$$g'^7 = 1 - g^{9/7},$$

and hence
$$\xi = \int_0^g \frac{dg}{(1-g^{9/7})^{1/7}}. \tag{6.17}$$

This integral was computed numerically, and at each stage the corresponding values for $f$, $f'$ and $\eta$ were calculated. The value of $\xi$ at the edge of the jet, $g = 1$, is expressible as a beta function, and thus a useful check was made on the accuracy of the work. The variations with $\eta$ of $f$ and $f'$ are illustrated in figure 2. The solution is satisfactory up to the velocity maximum, but the rapid fall of velocity in the outer region, a consequence of the assumption $\epsilon \propto u^6$, is quite unacceptable. A more realistic assumption is that $\epsilon$ is constant in the outer part of the jet, in accordance with Prandtl's hypothesis. We now consider how this may be achieved.

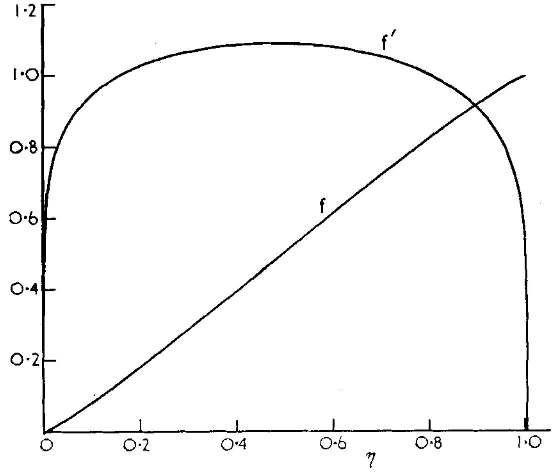


Figure 2. Turbulent wall jet, eddy viscosity proportional to $u^6$. Variation of mass flux ($f$) and velocity ($f'$) with distance from the wall ($\eta$).

## 7. Matching of solutions

The first thing to realize is that, if the inner part of the wall jet is to be governed by Blasius's formula, and the outer part by Prandtl's hypothesis, complete similarity is no longer possible. For, as functions of $x$, $\epsilon \propto R^{3/4}$ in the inner layer, as shown in (6.5), and $\epsilon \propto R$ in the outer layer, where $R = u\delta/\nu$ is the Reynolds number. With increasing $x$, $u$ decreases rather faster than $\delta$ increases, and hence $R$ decreases slowly. Consequently the velocity profile must change gradually as the jet develops, the inner layer occupying a progressively larger part of the jet width. This is not dependent on the precise forms selected for the eddy viscosity in the parts of the jet near and far from the wall. Any method which relates the inner part to the presence of the wall, and the outer part to free turbulent flow, will give the same conclusion.

However, the ratio of the eddy viscosities in the outer and inner parts of the layer varies only as the fourth root of the slowly changing Reynolds number, and we shall ignore this relative variation in seeking to derive a solution. A process of matching inner and outer solutions will then enable us to estimate the Reynolds number to which a particular matched profile is appropriate. It is hoped by this means to predict the main features of an actual turbulent wall jet. The assumptions in regard to the eddy viscosity, on which the whole method depends, are themselves of such dubious validity that it is hardly justifiable to try to develop any more precise procedure at this stage.

We now consider the equations for the inner and outer parts of the jet. For the inner layer, equations (6.5) to (6.11) continue to apply unchanged. In the outer layer we retain the same variables, but, in place of (6.8), we write

$$\epsilon = \lambda \bar{x}^{3-3b}\nu, \tag{7.1}$$

which is proportional to $R^{3/4}$, instead of being proportional to $R$ as indicated by Prandtl's hypothesis. This is the price that must be paid if we are still to have a solution in terms of the similarity variables. The equations are

$$\frac{d}{d\eta}(Af_1'^6 f'') + f_1 f_1'' + \alpha f_1'^2 = 0, \tag{7.2}$$

$$f_0''' + f_0 f_0'' + \alpha f_0'^2 = 0, \tag{7.3}$$

where the suffixes denote the inner and outer layers respectively. The values of $a$ and $b$, the similarity exponents, are given as functions of $\alpha$ by (6.11). As boundary conditions, we have $f_1'\eta^{-1/7} \to$ constant as $\eta \to 0$, and $f_0'(\infty) = 0$. Clearly certain further conditions must hold at the junction of the inner and outer layers. The value $\alpha = 2$ was determined in previous cases by exterior momentum flux arguments. These are no longer helpful, so $\alpha$ remains arbitrary. Indeed it is the variation of $\alpha$ along the jet length which represents the gradual change of jet profile, as different values of $\alpha$ will be appropriate at different Reynolds numbers.

Details of the calculations involved in the integration of (7.2) and (7.3), for general values of $\alpha$, are deferred to the next section. For the present, we merely assert that such solutions are available. In (7.2) it is convenient to write

$$f_2(\eta) = A^{1/5} f_1(\eta), \tag{7.4}$$

so that the equation becomes

$$\frac{d}{d\eta}(f_2'^6 f_2'') + f_2 f_2'' + \alpha f_2'^2 = 0. \tag{7.5}$$

Now if one solution of (7.5), with the correct behaviour near $\eta = 0$, has been found as $f_2(\eta)$, then, as in § 6, the general solution can be written as $X^7 f_2(X^{-5}\eta)$, for some constant $X$. The corresponding solution of (7.2) is

$$f_1(\eta) = A^{-1/5} X^7 f_2(X^{-5}\eta). \tag{7.6}$$

The conditions to be satisfied at the junction between the inner and outer layers will now be considered. Suppose that the join is to be at the velocity maximum, and here $\psi$ and $u$ are to be continuous. Thus at the velocity maximum $\eta = \eta_m$,

$$f_0'' = f_1'' = f_2'' = 0, \qquad f_0 = f_1 = A^{-1/5} X^7 f_2, \qquad f_0' = f_1' = A^{-1/5} X^2 f_2'. \tag{7.7}$$

Since the solutions $f_0$ and $f_2$ are known, the values of $A$ and $X$ can be found at once. The eddy viscosity is discontinuous at $\eta = \eta_m$, which is an apparent drawback. However, the shearing stress is continuous, being zero on each side of the join.

The eddy viscosity in the outer layer may now be determined as follows. Blasius's formula (6.2) shows that, near the wall,

$$\frac{\tau}{\rho} = \epsilon \frac{\partial u}{\partial y} = 0{\cdot}0225 u^2 \left(\frac{\nu}{uy}\right)^{1/4}. \tag{7.8}$$

With the expressions of § 6, this takes the form

$$A\lambda f_1'^4 f_1''(\eta f_1')^{1/4} = 0{\cdot}0225 \tag{7.9}$$

for small $\eta$. If $f_1'\eta^{-1/7} \to D$ as $\eta \to 0$, (7.9) gives

$$A\lambda D^{21/4} = 0{\cdot}1575, \tag{7.10}$$

and if $f_2'\eta^{-1/7} \to E$ as $\eta \to 0$, (7.6) shows that $D = A^{-1/5}X^{9/7}E$, and hence (7.10) becomes

$$\begin{aligned}\lambda &= 0{\cdot}1575A^{1/20}X^{-27/4}E^{-21/4}\\ &= 0{\cdot}1575E^{-21/4}\left(\frac{f_{2m}}{f_{0m}}\right)^{5/4}\frac{f_{0m}'}{f_{2m}'},\end{aligned} \tag{7.11}$$

from (7.6), where the suffix $m$ denotes the value at the velocity maximum. All the terms of the right-hand side of (7.11) are supposed known, for any $\alpha$, and thus $\lambda$ is determined. Now, according to Prandtl's hypothesis, in the outer layer,

$$\epsilon = \kappa u_m \delta_t, \tag{7.12}$$

where $u_m$ is the maximum velocity, $\delta_t$ a typical measure of the jet width, and $\kappa$ a universal constant; i.e. one which should not depend on $\alpha$. The jet Reynolds number may be defined, more precisely than previously, as $R = u_m\delta_t/\nu$, and it now follows from (6.7) and (7.1) that

$$\lambda = \kappa(\eta_t f_{0m}')^{3/4}R^{1/4}, \tag{7.13}$$

where $\eta_t$ is the value of $\eta$ corresponding to $\delta_t$. Hence, from (7.11),

$$\kappa R^{1/4} = 0{\cdot}1575E^{-21/4}\left(\frac{f_{2m}}{f_{0m}}\right)^{5/4}\frac{f_{0m}'}{f_{2m}'}(\eta_t f_{0m}')^{-3/4}. \tag{7.14}$$

## 8. Integrations and results

The integration of equations (7.3) and (7.5), for general values of $\alpha$, will now be considered. As in § 4, we may choose the solution of (7.3) for which $f_0(\infty) = 1$. A series expansion, valid for large values of $\eta$, is readily obtained in the form

$$\begin{aligned}f_0(\eta) = 1 - e^{-\eta} + \tfrac{1}{4}(1+\alpha)e^{-2\eta} - \tfrac{1}{72}(1+\alpha)(5+4\alpha)e^{-3\eta} + \\ + \tfrac{1}{1728}(1+\alpha)(34+53\alpha+21\alpha^2)e^{-4\eta} - \dots,\end{aligned} \tag{8.1}$$

where the coefficient of $e^{-\eta}$ has been arbitrarily taken as unity. This is equivalent to making a suitable choice of the zero of $\eta$. From the series, $f_0$ and its derivatives were calculated at a suitable set of values of $\eta$, and the solution of (7.3) was then extended to smaller values of $\eta$ by numerical integration, until the zero of $f_0''$ was passed. The values of $f_{0m}$, $f_{0m}'$ and $\eta_t$ were then estimated, $\eta_t$ being chosen as the interval between the point at which $f_0' = f_{0m}'$ and the point at which $f_0' = \frac{1}{2}f_{0m}'$. This procedure was carried out for $\alpha = 1{\cdot}1$, $1{\cdot}2$, $1{\cdot}4$, $1{\cdot}6$ and $1{\cdot}8$. The solution for $\alpha = 2$ was found in § 4, and, for $\alpha = 1$, equation (7.3) is that corresponding to a plane free jet, with solution $f_0 = \tanh\frac{1}{2}\eta$. The values of $f_{0m}$, $f_{0m}'$ and $\eta_t$ thus determined are shown in figure 3, and lie on smooth curves, considered as functions of $\alpha$. From this figure, the values corresponding to any required $\alpha$ in the range can be read off immediately, to sufficient accuracy.

A point which emerges from the integrations, is the great insensitivity to the value of $\alpha$ of the velocity profile in the region beyond the velocity maximum. In figure 4, where $f_0'/f_{0m}'$ is plotted as a function of $(\eta-\eta_m)/\eta_t$,

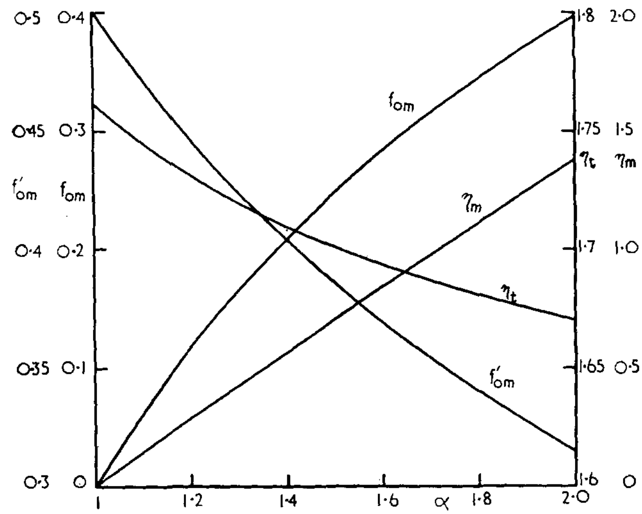


Figure 3. Results for a turbulent wall jet. Values of mass flux ($f_{om}$) and velocity ($f'_{om}$) at the velocity maximum. Also thickness of the inner layer ($\eta_m$) and outer layer ($\eta_t$).

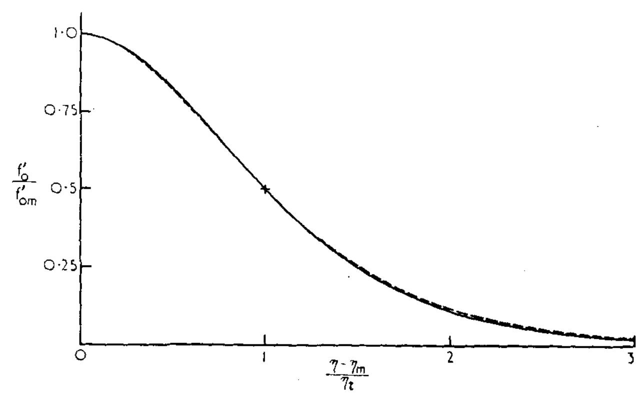


Figure 4. Comparison of velocity profiles. ——— $\alpha = 1$, ---- $\alpha = 2$.

the curves for the extreme values $\alpha = 1$ and $\alpha = 2$ are seen to be so close to each other that no experiment could be expected to differentiate between them. Thus the outer part of the velocity profile of a laminar wall jet is virtually the same as that of a plane free jet.

For the inner part of the wall jet, we wish to determine the solution of (7.5) for which $f_2'\eta^{-1/7} \to$ constant as $\eta \to 0$. This implies that $f_2 f_2''/f_2'^2 \to \frac{1}{8}$ as $\eta \to 0$. Also, at the velocity maximum, which is the farthest point to which the solution need be carried, $f_2'' = 0$. Thus the second term of (7.5) is relatively unimportant compared with the third, throughout the range of integration. This suggests that a satisfactory approximation might be obtained by replacing the term $f_2 f_2''$ in (7.5) by $\theta f_2'^2$, where $\theta$ is a small positive constant, not depending on $\alpha$. The equation then becomes

$$\frac{d}{d\eta}(f_2'^6 f_2'') + (\alpha+\theta) f_2'^2 = 0. \tag{8.2}$$

On writing $$f_3(\eta) = B f_2(\eta), \qquad B^5 = (2+\theta)/(\alpha+\theta), \tag{8.3}$$

(8.2) becomes $$\frac{d}{d\eta}(f_3'^6 f_3'') + (2+\theta) f_3'^2 = 0, \tag{8.4}$$

in which $\alpha$ does not appear. The reason for the particular choice of the substitution (8.3) is that, by reversing the arguments given above, equation (8.4) may be replaced by (7.5) with $\alpha = 2$. This is equation (6.13), of which the solution was found in § 6. Denoting this known solution by $f_3(\eta)$, we obtain the required solution of (7.5) from (8.3), once the value of $\theta$ has been chosen.

The accuracy of this approach can be judged by a consideration of the extreme case $\alpha = 1$. Equation (7.5) can in this case be integrated twice immediately, to give

$$\tfrac{1}{7} f_2'^7 + \tfrac{1}{2} f_2^2 = C\eta, \tag{8.5}$$

where $C$ is a constant. As previously, $C$ may be chosen arbitrarily, and the solution of (8.5) for a general value of $C$ is then expressible as $Y^7 f_2(Y^{-5}\eta)$ for some constant $Y$. With $C = \frac{1}{7}$, equation (8.5) was integrated by a method of successive approximations. The following results were obtained.

$$\left.\begin{array}{llll} \text{As} & \eta \to 0, & f_2'\eta^{-1/7} \to 1. & \\ \text{When}\quad f_3'' = 0, & f_2 = 0{\cdot}188, & f_2' = 0{\cdot}759, & \eta = 0{\cdot}269. \end{array}\right\} \tag{8.6}$$

These exact values may be compared with those we should deduce on the basis of the approximation developed above, using the relation $f_3(\eta) = BY^7 f_2(Y^{-5}\eta)$, and the following properties of $f_3(\eta)$ which were found in the integrations of § 6.

$$\left.\begin{array}{llll} \text{As} & \eta \to 0, & f_3'\eta^{-1/7} \to 1{\cdot}322. & \\ \text{When}\quad f_3'' = 0, & f_3 = 0{\cdot}479, & f_3' = 1{\cdot}091, & \eta = 0{\cdot}478. \end{array}\right\} \tag{8.7}$$

With a suitable choice of the two constants $B$ and $Y$, (8.7) should give values similar to those of (8.6) for the four quantities concerned. The value of $\eta$ agrees if $Y = 1{\cdot}121$, and the values of $B$ then required to give agreement of $f_2$, $f_2'$ and $\lim_{\eta \to 0}(f_2'\eta^{-1/7})$ are 1·140, 1·142 and 1·140 respectively. This is a remarkable justification of the ideas we have used. Equation (8.3) now shows that $\theta = 0{\cdot}07$, a very acceptable value in view of the arguments leading to (8.2).

Thus for the inner layer, as for the outer, the form of the velocity distribution is effectively independent of $\alpha$. However, the velocity distribution for the whole wall jet does vary with $\alpha$, due to changes in the relative scaling factors in the two parts. The gradient near the wall becomes steeper as $\alpha$ decreases. A typical velocity profile, that corresponding to $\alpha = 1{\cdot}4$, is shown in figure 5. The width of the inner layer

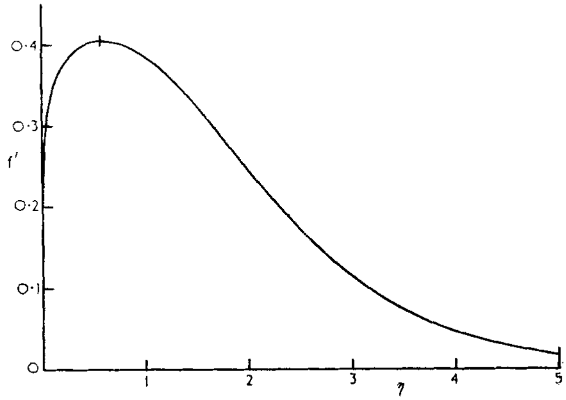


Figure 5. Velocity profile for a turbulent wall jet, $\alpha = 1{\cdot}4$.

is proportional to $\eta_m = 1{\cdot}09 f_{0m}/f'_{0m}$ and that of the outer layer to $\eta_t$, and so the velocity profiles corresponding to other values of $\alpha$ may readily be deduced from the curves of figure 3, where both $\eta_m$ and $\eta_t$ are given.

The results for the inner layer, obtained above, may now be inserted in (7.14) to give

$$\kappa R^{1/4} = \frac{0{\cdot}0275}{\alpha + 0{\cdot}07}(f_{0m})^{-5/4}(f'_{0m})^{1/4}\eta_t^{-3/4}, \tag{8.8}$$

and on substituting the values for the outer layer, from figure 3, we obtain the results shown in table 1.

| $\alpha$ | $\kappa R^{1/4}$ | $R$ ($\kappa = 0{\cdot}012$) | $a$ | $b$ |
|---|---|---|---|---|
| 1·1 | 0·415 | $1{\cdot}4 \times 10^6$ | −1·050 | 1·005 |
| 1·2 | 0·170 | $4{\cdot}1 \times 10^4$ | −1·096 | 1·010 |
| 1·3 | 0·102 | 5200 | −1·139 | 1·015 |
| 1·4 | 0·0707 | 1200 | −1·178 | 1·019 |
| 1·5 | 0·0530 | 380 | −1·214 | 1·023 |
| 1·6 | 0·0417 | 150 | −1·247 | 1·026 |
| 1·8 | 0·0287 | 33 | −1·307 | 1·033 |
| 2·0 | 0·0215 | 10 | −1·359 | 1·038 |

Table 1

The value of the constant $\kappa$ has to be determined empirically. Preliminary experiments (Bakke 1956), at a Reynolds number in the neighbourhood of 5000, recorded a velocity profile in good agreement with that predicted for $\alpha = 1{\cdot}3$. This indicates that $\kappa = 0{\cdot}012$ is an appropriate choice, and the corresponding values of $R$ are also shown in table 1. It may be noted that this value of $\kappa$ is approximately one-half of that found to be appropriate for a circular jet, and one-third of that for a plane free jet. More accurate experiments might indicate a modified value to be preferable. The extreme insensitivity of $\alpha$ to the value of the Reynolds number powerfully supports the assumption of approximate similarity, on which our whole theory has been based.

Summing up, table 1 represents our main conclusions in regard to the behaviour of a turbulent wall jet, and it may be hoped that these would not prove unduly sensitive to changes in our assumptions as to the nature of the turbulent shearing stress. The predicted velocity distribution is given by figure 5, with suitable scaling of the inner and outer layers as discussed above. The similarity exponents, specifying the ratio of change of the maximum velocity and the jet width, are given as functions of $\alpha$ by equation (6.11); a few comments on the accuracy of this equation are made in the next and final section.

## 9. Values of similarity exponents

The one purely arbitrary feature of the matching procedure developed in §7, was the decision to link the coordinate system to the inner layer, and to ignore variations in the outer layer. An equally valid alternative would be to choose coordinates appropriate for the outer layer, and to assume the skin-friction to have the value required by Blasius's formula at the particular Reynolds number. The whole analysis would go through as before, and identical velocity profiles would be deduced. In place of (6.11) we should have, as in §5,

$$a = -\frac{2\alpha}{1+\alpha}, \qquad b = 1, \tag{9.1}$$

as is easily verified. Thus if $\alpha = 1{\cdot}3$, (6.11) gives

$$a = -1{\cdot}15, \qquad b = 1{\cdot}03, \tag{9.2}$$

while (9.1) gives

$$a = -1{\cdot}13, \qquad b = 1. \tag{9.3}$$

Although the difference between (9.2) and (9.3) is not great, it would be satisfying to see how it arises.

For $\alpha = 1{\cdot}3$, equation (9.2) shows that $R \propto x^{-0{\cdot}12}$. If $R$ changes from that corresponding to $\alpha = 1{\cdot}3$ at $x = x_A$, to that corresponding to $\alpha = 1{\cdot}4$ at $x = x_B$, then approximately, from table 1,

$$\frac{x_B}{x_A} = \left(\frac{5200}{1200}\right)^{1/0{\cdot}12} \doteqdot 200\,000. \tag{9.4}$$

Incidentally, this figure strikingly illustrates the slowness of the change in velocity profile. Consider a wall jet which has developed from $x_A$ to $x_B$. On the theory of §7, its profile at $x_B$ is still that appropriate to $\alpha = 1{\cdot}3$. However, in fact the profile should be that corresponding to $\alpha = 1{\cdot}4$, with the scales of $f$ and $\eta$ suitably adjusted. If the maximum velocity and width of the outer layer are $u_{1\cdot 3}$ and $\delta_{1\cdot 3}$ for $\alpha = 1\cdot 3$, and $u_{1\cdot 4}$ and $\delta_{1\cdot 4}$ for $\alpha = 1\cdot 4$, then if the adjustment is made so that the mass flux, and also the velocity gradient in the inner layer (for which the solution is known to be accurate), are unchanged, a rough calculation shows that, approximately,

$$\left.\begin{aligned} u_{1\cdot 4}/u_{1\cdot 3} &= 1\cdot 12 \doteqdot (200\,000)^{0\cdot 01}, \\ \delta_{1\cdot 4}/\delta_{1\cdot 3} &= 0\cdot 84 \doteqdot (200\,000)^{-0\cdot 015}. \end{aligned}\right\} \tag{9.5}$$

Adding these contributions from the changing profile to the values of (9.2), we have as revised estimates of the similarity exponents

$$a = -1\cdot 14, \qquad b = 1\cdot 015. \tag{9.6}$$

The fact that these figures are precisely intermediate between (9.2) and (9.3) must be considered entirely fortuitous, in view of the roughness of the calculations, but the order of magnitude of the differences arising from the alternative choice of coordinates is satisfactorily explained. Physically, we may argue from (7.1), that, with $\alpha$ constant, the velocity gradient in the outer layer becomes progressively too small as $x$ increases, and hence an adjustment qualitatively similar to (9.5) is required. For the alternative approach, leading to (9.1), the velocity gradient in the inner layer becomes progressively too steep as $x$ increases, and so necessitates an adjustment in the opposite direction.

The discrepancies between (6.11) and (9.1) are probably no larger than other errors inherent in the whole method of analysis, but in practice the mean of the two might well be adopted as the best prediction of our theory. These mean values are also given in table 1.

For a plane turbulent wall jet, the analysis of the last sections all applies unchanged, with appropriately modified values of the similarity exponents. Equation (6.11) is replaced by

$$a = -\frac{4\alpha}{5+4\alpha}, \qquad b = \frac{4+4\alpha}{5+4\alpha}, \tag{9.7}$$

and (9.1) is replaced by

$$a = -\frac{\alpha}{1+\alpha}, \qquad b = 1. \tag{9.8}$$

The value of the constant $\kappa$ must be determined afresh, as there is no reason to suppose that it still has the same value as that for a radial wall jet.

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